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Abstract

The steady, zonally symmetric motion of a shallow incompressible atmosphere on a rapidly rotating earth subject to an equator-ward temperature gradient is studied. The assumptions made allow the thermodynamics to be treated separately from the motions. The turbulence terms are modelled using mixing length arguments. Assuming the turbulence length scale is small compared to the earth's radius and the motion is slow compared to the earth's rotation speed, the north-south geopotential gradient drives the eastward winds in the "interior" temperate regions, that is, not near the surface, equator or poles. The meridional winds in the interior are driven by the turbulence generated by the shear in the eastward winds. Near the equator the advective terms become comparable with the rotational terms, but the turbulence terms remain unimportant. The motions there show trade winds at the equator, changing to eastward at a predicted latitude of about 15°. The meridional motion takes the form of Hadley cells with rising at the equator and sinking again at about 21°. The Hadley cell and the temperate region are connected through a vertical layer of turbulence. Near the poles the advective and turbulent terms become comparable with the rotation terms. Surface Ekman layers complete the picture.

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1. Introduction

When viewed from outer space, the earth's atmosphere, as evidenced by the cloud patterns, shows no signs of symmetry or regularity. However, analysis of wind and temperature data over long periods of time shows that on the average the atmosphere behaves in a regular manner. Prevailing westerlies, trade winds and the Hadley circulation are gross features of the motion which are observed from average data.

In 1735, George Hadley initiated the idea that solar heating at the equator forced the air there to rise, and hence he conjectured that the air, once aloft, travelled to the poles, where it sank back to the surface to journey toward the equator again. Subsequent observations by Ferrel around the 1880's suggested that the air rising at the equator did not travel to the poles, but sank back to the surface at about 30° north or south latitude. In addition, the cold air sinking at the poles rose again at about 60° north or south latitude. In between these cells was a single cell rising at 60° and sinking at 30° in each hemisphere. The conjectures mechanism to drive the temperature cell was friction between the adjacent thermally driven cells; while the eastward winds therein were the result of the rotational (Coriolis) force interacting with the meridional motions. Because of the appearance of three cells in each hemisphere, mechanisms to explain the motions by explaining the presence of each cell have been called tricellular theories.

The aforementioned tricellular theory of driving the motions has fallen into disfavor, one reason being that if the heating drives the motions in the tropical and polar cells, the winds in the temperate cell are necessarily weaker since it is driven by the others. But the prevailing westerlies observed in the temperate cell are much too strong to be secondary or frictionally forced winds.

A recent popular explanation of the driving mechanism is the wave theory (Pfeffer, 1964). In this theory, the north-south temperature gradient induced by solar heating drives wave cyclones and anticyclones. The nonlinear interactions of these waves cause a mean zonal flow. The zonal flow in turn drives the meridional cells through the mechanism of the Coriolis force.

We give a new explanation for the general circulation in which the north-south temperature gradient drives the zonal motions in the temperate cell through the geopotential gradient. The zonal motions in the tropical and polar cells are driven by the zonal motion in the temperate cell through friction between the cells. More precisely, the eastward motion at the poleward edge of the tropical cell causes an eastward motion inside the tropical cell, at least for some distance. The Coriolis effects cause a strong meridional motion and the trade winds. The meridional motions in the temperate cell are driven by frictional interactions with the mean zonal flow and the tropical and polar meridional motions.

In order to describe the motions, we shall use the fluid mechanical and thermodynamical equations for the zonally symmetric (i.e. independent of longitude) flow of a shallow layer of fluid. Moreover, following Saltzman (1968), we shall make a

Boussinesq-like approximation, neglecting density variations except in the vertical momentum equation. The shallow atmosphere assumption allows us to use the hydrostatic pressure equation for vertical momentum balance. This assumption also allows the neglect of the vertical velocity, except in terms involving vertical derivatives, which are large. The hydrostatic pressure equation allows us to use the pressure as the vertical coordinate, replacing $\hat{z}$, the height above sea level. The relevant vertical "velocity" is $\hat{\omega} = d\hat{p}/dt$, the material derivative of the pressure. We shall refer to $\hat{\omega}$ as the vertical $\hat{p}$-velocity, and to $\hat{w} = d\hat{z}/dt$ as the vertical $\hat{z}$-velocity. The pressure gradient terms are expressed in terms of the geopotential, defined as $\hat{\Phi} = g\hat{z}(\hat{\phi},\hat{p})$, where $\hat{z}(\hat{\phi},\hat{p})$ is the height of the isobar surface of pressure $\hat{p}$.

The average motions which we wish to describe are, to a large degree, both steady and zonally symmetric. Thus we shall assume that time and longitudinal derivatives may be neglected.

When the equations are appropriately nondimensionalized, there appear two important parameters, the Rossby number $Ro = U/2\Omega a$, and the Ekman number $\mathcal{E} = \Theta/2\Omega U$. Here U is the velocity scale, Ω is the rotation rate of the earth, a is the radius of the earth, and Θ is the scale of the turbulent flux of momentum per unit mass. In the earth's atmosphere, the Ekman number and the Rossby number are both small, typical values being $\mathcal{E} \cong 10^{-3}$, $Ro \cong 10^{-1}$. The presence of these small parameters in the problem allows the hope of meaningful approximate solutions valid in different parts of the meridional $(\hat{\phi} - \hat{p})$ plane.

For example, the Ekman number, which measures the importance of turbulence, is quite small, and presumably multiplies the highest derivatives. This implies that the regions where turbulence is important are boundary layers which are thin compared to the scale of global motions. The Rossby number, which measures the importance of advection, is also small, although not as small as the Ekman number. Thus the advection terms are important only in relatively small regions compared to large scale motions. The regions where advection is important, however, are large compared to the turbulent or Ekman layers.

2. Equations of Motion

The equations of motion and heat balance for the zonally symmetric flow of a shallow atmosphere (Saltzman, 1968) are

$$\hat{\omega}\frac{\partial\hat{u}}{\partial\hat{p}} - \left(\hat{f} - \frac{\partial\hat{u}\cos\hat{\phi}}{a\cos\hat{\phi}\,\partial\hat{\phi}}\right)\hat{v} - \hat{X} = 0\,, \tag{2.1}$$

$$\hat{v}\frac{\partial\hat{v}}{a\partial\hat{\phi}} + \hat{\omega}\frac{\partial\hat{v}}{\partial\hat{p}} + \left(\hat{f} + \frac{\tan\hat{\phi}}{a}\hat{u}\right)\hat{u} + \frac{\partial\hat{\Phi}}{a\partial\hat{\phi}} - \hat{Y} = 0\,, \tag{2.2}$$

$$\frac{\partial\hat{\Phi}}{\partial\hat{p}} + \frac{R}{\hat{p}}\hat{T} = 0\,, \tag{2.3}$$

$$\frac{\partial\hat{\omega}}{\partial\hat{p}} + \frac{\partial\hat{v}\cos\hat{\phi}}{a\cos\hat{\phi}\,\partial\hat{\phi}} = 0\,, \tag{2.4}$$

$$\hat{v}\frac{\partial\hat{T}}{a\partial\hat{\phi}} + \left(\frac{\partial\hat{T}}{\partial\hat{p}} - \frac{R\hat{T}}{c_p\hat{p}}\right)\hat{\omega} - \hat{Q} = 0\,. \tag{2.5}$$

A caret is used to denote dimensional variables. Here $\hat{u}$, $\hat{v}$ are the eastward and northward components of velocity, $\hat{\phi}$ is the latitude, $\hat{p}$ is the pressure, $\hat{\omega} = d\hat{p}/dt$ (t is time), $\hat{\Phi}$ is the gravitational potential, a is the planet radius, $\hat{T}$ is the temperature, R is the gas constant, and c_p is the heat capacity of the atmosphere. Also, $\hat{f} = 2\Omega \sin \hat{\phi}$ is the Coriolis parameter, where Ω is the angular velocity of planetary rotation. The quantities $\hat{X}$, $\hat{Y}$, $\hat{Q}$, to be specified, represent turbulent diffusion and thermal forcing of the atmospheric motions.

It is not our goal to study the thermodynamics of the atmosphere. We shall assume that we can prescribe the gravitational potential $\hat{\Phi}(\hat{\phi},\hat{p})$. This assumption, along with the Boussinesq approximation, effectively separates the thermodynamics from the mechanics, since by determining the motions in terms of $\hat{\Phi}$, we can eliminate the velocity components from the thermodynamic equations, yielding equations for $\hat{\Phi}$, $\hat{T}$ and other appropriate variables. Saltzman (1968) discusses a similar approach.

The region in space where these equations govern the motion is

$$-\frac{\pi}{2} \leq \hat{\phi} \leq \frac{\pi}{2}$$

$$0 \leq \hat{p} \leq p_s(\hat{\phi}) ,$$

where $p_s(\hat{\phi})$ is the pressure at the surface.

The boundary conditions which are needed to complete the system fall into two categories, turbulent (viscous) and symmetry. The turbulent boundary conditions are

(2.6a,b) $\hat{u} = \hat{v} = 0$ on $\hat{p} = p_s(\hat{\phi})$,

(2.7) no stress on $\hat{p} = 0$.

Because the level $\hat{p} = 0$ actually corresponds to $z = \infty$, an unrealizable situation, we shall not attempt to satisfy (2.7). This boundary condition can presumably be satisfied by inclusion of a turbulent boundary layer at $\hat{p} = 0$. A more realistic approach to the boundary condition at the "top" of the atmosphere might be to apply a condition at some $\hat{p} = p_1 << p_s(\hat{\imath})$ corresponding to a shallow atmosphere, say at about 50 km. The appropriate condition should reflect the presence of the atmosphere above 50 km.

The symmetry boundary conditions are

(2.8a,b) $\hat{v} = \frac{\partial\hat{u}}{\partial\hat{\phi}} = 0$ at $\hat{\phi} = 0$,

(2.9a,b) $\hat{v} = \hat{u} = 0$ at $\hat{\phi} = \pm\frac{\pi}{2}$.

Let us nondimensionalize the equations with

(2.10)
$$\left\{\begin{array}{l} \hat{u} = Uu , \\ \hat{v} = Vv , \\ \hat{p} = p_o p , \quad \hat{\phi} = \phi_c + \phi_o \phi , \\ \hat{\omega} = (p_o V/a\phi_o)\omega , \\ \hat{\Phi} = \Phi_o \Phi , \\ (\hat{X},\hat{Y}) = \Theta(X,Y) , \end{array}\right.$$

where U is a velocity scale, ϕ_o is the latitudinal scale of the motion, p_o is a pressure scale, taken to be the surface pressure at the equator. The scale of $\hat{\omega}$, $p_o U/a\phi_o$, is suggested by the continuity equation (4). We also assume that Φ_o is an appropriate scale for the gravitational potential, and Θ is a scale for the turbulent stresses.

We shall denote $\Theta/2\Omega U$ by ξ, and $U/2\Omega a$ by Ro. We call these two parameters the Ekman number and the Rossby number, respectively. It is important to note that the Rossby number defined here is not that used by Charney (1948, 1963). He includes the quantity sin $\hat{\phi}$ in his Rossby number. The Rossby number we have defined is a constant, independent of $\hat{\phi}$.

In terms of these parameters the nondimensionalized equations become

$$(2.11) \quad \frac{\alpha Ro}{\phi_o}\left(v \frac{\partial u \cos(\phi_c + \phi_o\phi)}{\cos(\phi_c + \phi_o\phi)\partial\phi} + \omega \frac{\partial u}{\partial p}\right) - \alpha \sin(\phi_c + \phi_o\phi)v = \xi X ,$$

$$(2.12) \quad \frac{\alpha^2 Ro}{\phi_o}\left(v \frac{\partial v}{\partial \phi} + \omega \frac{\partial v}{\partial p}\right) + Ro \tan(\phi_c + \phi_o\phi)u^2 + \sin(\phi_c + \phi_o\phi)u$$

$$= \xi Y - \frac{\Phi_o}{2\Omega U a\phi_o} \frac{\partial \Phi}{\partial \phi}(\phi_c + \phi_o\phi, p) ,$$

$$(2.13) \quad \frac{\partial \omega}{\partial p} + \frac{\partial v \cos(\phi_c + \phi_o\phi)}{\cos(\phi_c + \phi_o\phi)\partial\phi} = 0 .$$

3. Turbulence Model

Let us now consider models for the turbulence terms. The northward transport of eastward momentum due to turbulence is

$$\hat{X}_N = - \frac{1}{a \cos \hat{\phi}} \frac{\partial}{\partial \hat{\phi}} \{ (\overline{\hat{v}'\hat{u}'}) \cos \hat{\phi} \} , \tag{3.1}$$

where a prime denotes the fluctuation from the zonally averaged flow, and the bar denotes the zonal average. We shall assume a mixing length hypothesis based on the assumption that differences in the eastward velocity are most effective in supporting the turbulence. For a discussion of mixing length theory, see Hinze (1959).

We assume that

$$\hat{u}' = \frac{\ell_1}{a} \frac{\partial \hat{u}}{\partial \hat{\phi}} , \tag{3.2a}$$

and

$$\hat{v}' = \frac{\ell_2}{a} \frac{\partial \hat{u}}{\partial \hat{\phi}} \tag{3.2b}$$

where ℓ_1 and ℓ_2 are mixing lengths. The northward transport of eastward momentum due to turbulence is

$$\hat{X}_N = \frac{L_H^2 U^2}{a^3 \phi_o^3} \frac{1}{\cos \hat{\phi}} \frac{\partial}{\partial \phi} \{ \lambda_H^2 \frac{\partial u}{\partial \phi} \left| \frac{\partial u}{\partial \phi} \right| \cos \hat{\phi} \} \tag{3.3}$$

where L_H is the horizontal size scale of the eddies, and λ_H is an average dimensionless mixing length, defined by

$$L_H^2 \lambda_H^2 = | \overline{\ell_1 \ell_2} | . \tag{3.4}$$

The appearance of the minus sign is accounted for by the following argument. If $\partial \hat{u}/\partial \hat{\phi} > 0$ and $\ell_1 > 0$, then $\hat{u}' > 0$; but the average momentum transport is southward, since we are moving a parcel with excess momentum $\frac{\ell_1}{2} \frac{\partial \hat{u}}{\partial \hat{\phi}}$ southward, so that $\hat{v}' < 0$. Similar arguments for $\partial \hat{u}/\partial \hat{\phi} < 0$ lead to (3.3).

The vertical transport of eastward momentum is

$$\hat{X}_V = -\frac{\partial}{\partial\hat{p}}\overline{(\hat{\omega}'\hat{u}')} \,. \tag{3.5}$$

If we again use mixing lengths, we have

$$\hat{u}' = \ell_3 \frac{\partial\hat{u}}{\partial\hat{p}} \tag{3.6a}$$

and

$$\hat{\omega} = \hat{\rho} g \ell_4 \frac{\partial\hat{u}}{\partial\hat{p}} \,, \tag{3.6b}$$

where we now assume that the velocity differences which are effective in the vertical momentum transport are vertical differences, so that the relevant derivative is $\partial\hat{u}/\partial\hat{p}$. Here ℓ_3 and ℓ_4 are effective pressure differences which scale the vertical turbulent eddies. The quantity $\hat{\rho}g$ converts vertical $\hat{z}$-velocity to vertical $\hat{p}$-velocity. This hypothesis leads to the following expression for the vertical transport of eastward momentum due to turbulence:

$$\hat{X}_V = \frac{L_V^2 \rho_o g U^2}{P_o^3} \frac{\partial}{\partial p} \{\rho\lambda_V^2 (\frac{\partial u}{\partial p})^2\} \,. \tag{3.7}$$

Here L_V is the vertical size scale of the eddies and λ_V is an average dimensionless mixing length, ρ_o is the density at the surface, and $\hat{\rho} = \rho_o\rho$.

For the turbulent transport of northward momentum, we shall use similar considerations. Again we shall assume that differences in <u>eastward</u> velocity are most effective in supporting the turbulence. This leads to the expression

$$(3.8) \qquad \hat{Y}_N = \frac{L_H^2 U^2}{a^3 \phi_o^3} \frac{\partial}{\partial \phi} \{\mu_H^2 (\frac{\partial u}{\partial \phi})^2\}$$

and

$$(3.9) \qquad \hat{Y}_V = \frac{L_V^2 U^2}{p_o^2 D} \frac{\partial}{\partial p} \{\mu_V^2 (\frac{\partial u}{\partial p})^2\}$$

where μ_H and μ_V are average dimensionless mixing lengths. We shall assume that λ_H, λ_V, μ_H and μ_V are constants. Then, without loss of generality, we can take $\lambda_H = 1$.

The corr^^t scale Θ for the turbulence terms is the larger of the vertical and horizontal scales,

$$(3.10) \qquad \Theta = \max \left\{ \frac{L_V^2 \rho_o g U^2}{p_o^3}, \frac{L_H^2 U^2}{a^3 \phi_o^3} \right\} .$$

By taking L_H = 100 km, and L_V = 100 mb, corresponding to a vertical eddy scale of a couple of kilometers, we find that the horizontal and vertical turbulence terms are comparable if $\phi_o = 1$. If ϕ_o is smaller, the horizontal terms are dominant. Unless otherwise specified, we shall use the horizontal turbulence terms for scaling purposes. We denote L_H/a by ε. We now assume that $\varepsilon << 1$ and $Ro << 1$. Since $\mathcal{E} = \frac{\varepsilon^2}{\phi_o^2} \cdot \frac{Ro}{\phi_o}$, the turbulence terms will be negligible compared to the advective terms as long as $\phi_o >> \varepsilon$. Thus for latitudinal scales larger than a typical large scale eddy, we can neglect the turbulence terms. More precisely, the role of turbulence has been relegated to various boundary layers in the flow.

4. The Interior or Temperate Approximation

In seeking approximate solutions, it is natural to assume $\phi_o = 1$, $\phi_c = 0$, and $\alpha = 1$ and neglect terms of order Ro. This yields the approximation

$$(4.1) \qquad -\sin\phi v = 0 ,$$

$$(4.2) \qquad \sin\phi u = -\partial\Phi/\partial\phi ,$$

$$(4.3) \qquad \frac{\partial\omega}{\partial p} + \frac{\partial v \cos\phi}{\cos\phi\, \partial\phi} = 0 .$$

Here we have set $\Phi_o = 2\Omega Ua$.

Equation (4.1) suggests not that $v \equiv 0$, but that the first term in the expansion of v is small compared to 1. An Ekman layer expansion for the surface boundary layer (Pedlosky, 1969) suggests that the next term in the expansion is of order $\mathcal{E}^{1/2}$. Pedlosky ignores the inertia terms and uses a viscous term to model the effects of turbulence. Inclusion of the inertia terms does not change the result. However, at this order in the inner equation v is still zero.

The appropriate balance in the inner region is $\alpha = \mathcal{E}$, so that

$$(4.4) \qquad -\sin\hat{\phi} v = X .$$

From equations (4.1 - 4.4), the mechanism driving the motion in the interior (temperate) region is as follows: the geo-potential drives the eastward motion through the Coriolis force. The meridional motions v and ω are driven by friction; more specifically, the turbulent stress in the eastward direction drives the northward motion. The vertical motion is determined through the continuity equation.

In order to determine the motions further, it is necessary to have expressions for the geopotential Φ and the density ρ, which appears in the eastward turbulent stress force X. Thermodynamic considerations are important in each expression.

Since the velocity scale U can be taken to be the largest value assumed by the magnitude of the velocity vector, it is approximately (in dimensional variables)

$$(4.5) \qquad U = \max \left| \frac{\partial\hat{\Phi}/\partial\hat{\phi}}{2\Omega a \sin \hat{\phi}} \right| .$$

If we take values of $\hat{\Phi}$ from Schutz and Gates (1971) the velocity scale derived from the model is about 30 m/sec, which gives a Rossby number of about 0.03. The eastward velocity in the northern hemisphere, as calculated from the data of Schutz and gates, is shown in Figure 1.

From the meridional motions, using the mixing length hypothesis of Section 3, we have

$$(4.6) \qquad v = -\frac{1}{\sin\phi}\left\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left[\frac{\partial u}{\partial\phi}\left|\frac{\partial u}{\partial\phi}\right|\cos\phi\right] + \beta\frac{\partial}{\partial p}\left(\frac{\partial u}{\partial p}\right)^2\right\},$$

where $\beta = \frac{L_V^2 \lambda_V^2 a^3 \rho_o g}{p_o^3 L_H^2}$ is assumed to be of order 1.

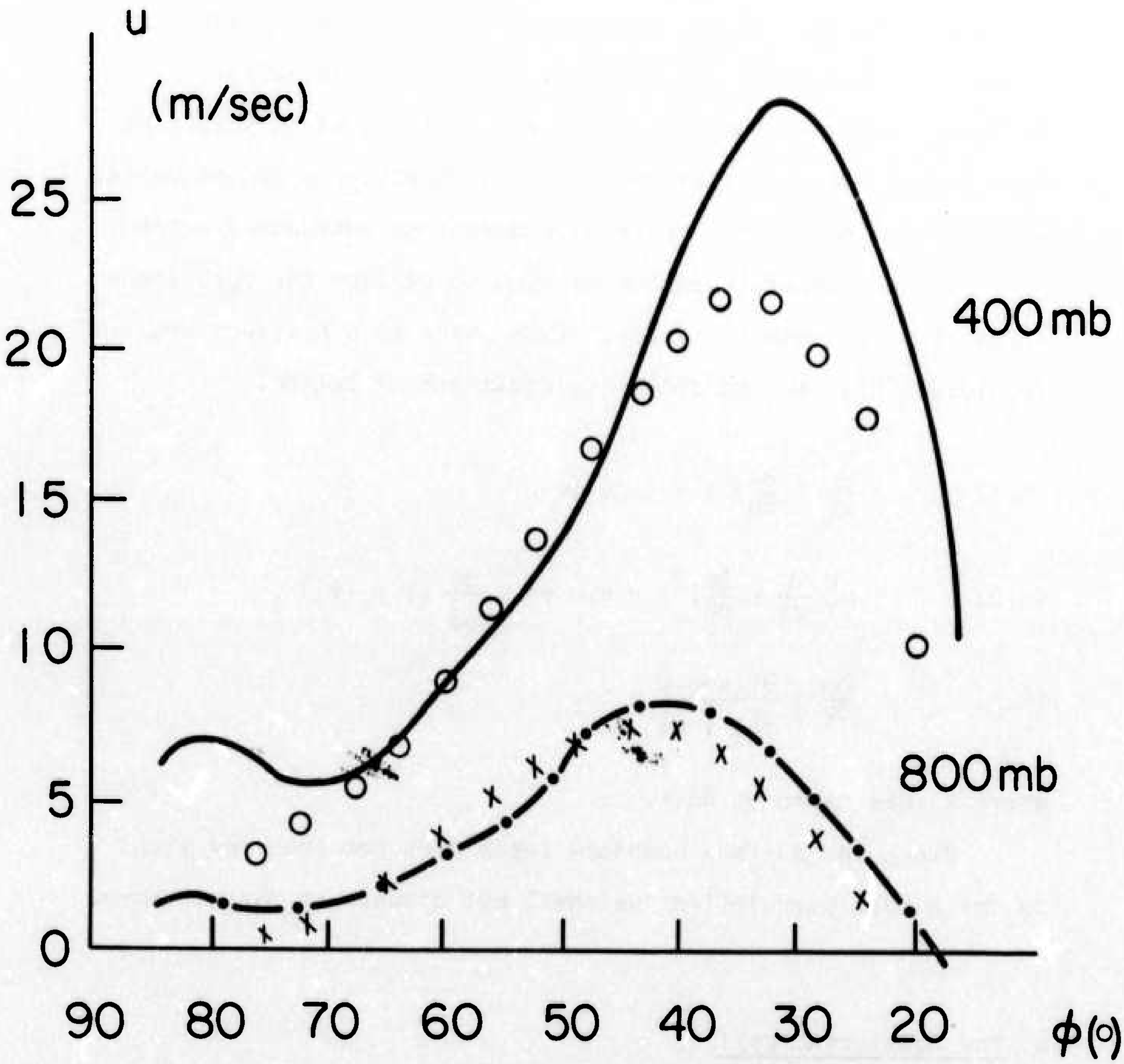

Figure 1. Zonal wind versus latitude:
at 400 mb —— predicted; o observed;
at 800 mb -.- predicted; x observed.

5. The Temperate Surface Boundary Layer

There are several boundary layers needed to complete the picture. It is well known that an Ekman (turbulent) boundary layer is needed to satisfy the boundary conditions at the earth's surface (2.6a,b). In the temperate region, where the advective terms are negligible, the analysis of Pedlosky is valid. Pedlosky, however, uses Newtonian viscosity terms to model the turbulence. We feel that the effect of turbulence can be better modelled by the mixing length considerations introduced before.

In the temperate region we wish to balance the turbulence terms with the rotation terms. This leads to a pressure scale of $(\rho_o g L_V^2 \frac{U}{2\Omega})^{1/3}$, and the following equations of motion.

$$\frac{\partial}{\partial p}\left[\left(\frac{\partial u}{\partial p}\right)^2\right] = -\sin\hat{\phi}v \,, \tag{5.1}$$

$$\mu_V^2 \frac{\partial}{\partial p}\left[\left(\frac{\partial u}{\partial p}\right)^2\right] = \sin\hat{\phi}u + \frac{\partial\Phi}{\partial\hat{\phi}}(\hat{\phi}, p_s(\hat{\phi})) \,, \tag{5.2}$$

$$\frac{\partial\omega}{\partial p} + \frac{\partial v\cos\hat{\phi}}{\cos\hat{\phi}\,\partial\hat{\phi}} = 0 \,, \tag{5.3}$$

where λ_V was taken to be 1.

Since the surface boundary layer does not shed any light on the global circulation, we shall not discuss it further here.

6. The Equatorial Cell

The approximation $\phi_o = 1$ is not valid near the equator, where $\hat{\phi}$ is small, or near the poles, where $\tan\hat{\phi}$ is large. To find solutions valid in these regions, we must re-scale the latitude by choosing ϕ_o to be small. The result is an advective "boundary layer", which is small compared to interior sizes, but large compared to the Ekman layers.

To obtain an approximation valid near the equator, we let $\phi_o = (Ro)^{1/2}$, $\phi_c = 0$, and $\alpha = 1$. The equations then become

$$(6.1) \qquad \frac{\partial \omega}{\partial p} + \frac{\partial v}{\partial \phi} = O(Ro^{1/2}) \ ,$$

$$(6.2) \qquad Ro \left(v \frac{\partial v}{\partial \phi} + \omega \frac{\partial v}{\partial p} + \phi u + \left. \frac{\partial \Phi}{\partial \phi} \right|_{\phi=0} \right) = O(Ro^{3/2}) \ ,$$

$$(6.3) \qquad Ro \, (v \frac{\partial u}{\partial \phi} + \omega \frac{\partial u}{\partial p} - \phi v) = O(Ro^{3/2}) \ .$$

In this case we scale the geopotential by $\Phi_o = (Ro)^{1/2}(2U\Omega)$. Thus, to lowest order in Ro we have

$$(6.4) \qquad \frac{\partial v}{\partial \phi} + \frac{\partial \omega}{\partial p} = 0 \ ,$$

$$(6.5) \qquad v \frac{\partial u}{\partial \phi} + \omega \frac{\partial u}{\partial p} = \phi v \ ,$$

$$(6.6) \qquad v \frac{\partial v}{\partial \phi} + \omega \frac{\partial v}{\partial p} = -\phi u - \left. \frac{\partial \Phi}{\partial \phi} \right|_{\phi=0} .$$

In terms of the equatorial dimensionless variables, the region on which we should consider (6.4 - 6.6) is

$$(6.7) \qquad -\infty < \phi < \infty \ , \quad 0 \leq p \leq 1 \ .$$

Let us attempt a solution to the equations of motion (6.4 - 6.6) under the conditions corresponding to "spring" or "fall", that is, symmetric forcing. If Φ is symmetric, then $\partial\Phi/\partial\phi\big|_{\phi=0} = 0$, and the equations become

$$v \frac{\partial u}{\partial \phi} + \omega \frac{\partial u}{\partial p} = \phi v \;, \tag{6.8}$$

$$v \frac{\partial v}{\partial \phi} + \omega \frac{\partial v}{\partial p} = -\phi u \;, \tag{6.9}$$

$$\frac{\partial v}{\partial \phi} + \frac{\partial \omega}{\partial p} = 0 \;. \tag{6.10}$$

The relevant boundary conditions are (2.8a,b). A turbulent surface boundary layer is needed to satisfy (2.6a,b).

The balance in the equatorial region, as expressed by equations (6.8 - 6.10) is the balance between rotation and advection. The driving mechanism (source of energy) is the boundary layer surrounding the equatorial region, with the important driving being through a turbulent layer between the equatorial region and the interior or temperate region.

The continuity equation (6.10) implies that there exists a stream function ψ with

$$v = \frac{\partial \psi}{\partial p} \quad \text{and} \quad \omega = -\frac{\partial \psi}{\partial \phi} \;. \tag{6.11}$$

Since we assume that there is no flow through the top ($p = 0$) or the bottom ($p = 1$) of the atmosphere, the lines $p = 0$ or 1 are streamlines.

The system of partial differential equations (6.8 - 6.10) is hyperbolic; hence there exist characteristic curves. The equations for the characteristics are

$$\frac{d\phi}{dt} = v \,, \tag{6.12}$$

$$\frac{dp}{dt} = \omega \,, \tag{6.13}$$

$$\frac{du}{dt} = \phi v \,, \tag{6.14}$$

$$\frac{dv}{dt} = -\phi u \,, \tag{6.15}$$

where t is a variable along a characteristic.

We observe from this system that $d\psi/dt = 0$ so that the characteristics for the system are the streamlines.

From the two momentum equations we see that the horizontal kinetic energy $E = \frac{u^2 + v^2}{2}$ is constant on each characteristic. We shall use E to parameterize the characteristics. This implies that $\psi = \psi(E)$.

The eastward momentum equation (6.14) along with (6.12) give $u = \frac{\phi^2}{2} + u_o(E)$. We then obtain v from $u^2 + v^2 = 2E$, giving

$$\frac{d\phi}{dt} = v = \pm\sqrt{2E - \frac{\phi^4}{4} - u_o\phi^2 - u_o^2} \,. \tag{6.16}$$

The solutions to the ϕ equation (6.16) depend on the sign of u_o. In order to get north-south symmetry, we must choose $u_o < 0$. This gives $v = 0$ at $\phi = 0$, and hence allows a convergence zone at the equator. The resulting solution for the ϕ equation (6.16) is

$$\phi(t) = \pm\phi_o(E)\,\mathrm{dn}\,(\frac{\phi_o t}{2}, k) \tag{6.17}$$

where dn is the delta amplitude function, and ϕ_o is related to u_o by

$$(6.18)\qquad \begin{cases} \dfrac{\phi_o^2}{2} + u_o \quad ^2 = 2E\ , \\ \text{and} \\ k = \dfrac{2\sqrt{2E}}{\phi_o^2} \quad ^{1/2} . \end{cases}$$

It remains to determine the location of the streamlines. To do this we write

$$(6.19)\qquad p = \int_{p_o(\phi)}^{p} dp + p_o(\phi)\ ,$$

for each ϕ and

$$1 = \frac{\partial\psi/\partial p}{v} = \frac{(d\psi/dE)}{v}\,(\partial E/\partial p)\ .$$

We shall assume that at $p = p_o(\phi)$, $v = 0$. Thus

$$(6.20)\qquad p = \int_{E_o(\phi)}^{E} \frac{d\psi/dE'}{\pm\sqrt{2E' - u_o^2 - u_o\phi^2 - \frac{\phi^4}{4}}}\, dE' + p_o(\phi)$$

where

$$(6.21)\qquad 2E_o(\phi) = \left(u_o(E_o(\phi)) + \frac{\phi^2}{2}\right)^2 .$$

We now impose the conditions that $p = 0$ and $p = 1$ are characteristics. We note that since $\phi = 0$ lies on both these curves, the value of E corresponding to these curves is given by

$$(6.22)\qquad 2E_1 = (u_o(E_1))^2\ .$$

This yields

$$(6.23)\qquad p_o(\phi) = 1/2$$

and

$$(6.24) \qquad \frac{1}{2} = \int_{E_o(\phi)}^{E_1} \frac{d\psi/dE'}{\sqrt{2E' - u_o^2 - u_o\phi^2 - \frac{\phi^4}{4}}}\, dE' \,.$$

This integral equation gives a relation between $\psi(E)$ and $u_o(E)$. Thus for this system, on the domain given previously, there is one arbitrary function, which we may take to be $u_o(E)$.

The appearance of an undetermined function in the solution is a common occurrence in inviscid flows. Presumably the function $u_o(E)$ can be determined by including dissipative effects, as in Batchelor (1956). The condition which $u_o(E)$ must satisfy is

$$\oint_{E=\text{const}} X\,dt = 0 \,,$$

where $X = \frac{\partial}{\partial\phi}\left(\frac{\partial u}{\partial\phi}\right)^2$. This leads to the following functional differential equation for u_o:

$$(6.25) \qquad Au_o'' + Bu_o'u_o'' + Cu_o'^2 + Du_o' + F = 0 \,,$$

where a prime denotes differentiation with respect to E, and

$$A = \oint \phi\omega^2 dt/\psi'^2 \,,$$

$$B = \oint \omega^3 dt \,,$$

$$C = \oint \omega\omega_\phi/\psi'^2 + \oint \omega^3 dt\psi''/\psi'^4 - \oint \phi\omega^2 dt\psi''/\psi'^3 \,,$$

$$D = -\oint \phi\omega_\phi dt/\psi' \,,$$

$$F = \oint \phi dt \,,$$

are functionals of u_o. Since the equation is second order, we need two conditions on $u_o(E_1)$. These will be discussed in Section 7.

We shall not attempt to solve equation (6.25) here. Instead, let us examine the motions obtained by taking u_o constant. One implication of this assumption is that the eastward velocity is then independent of the vertical coordinate. This assumption is somewhat unreal physically, but allows us to solve the integral equation (6.24).

With u_o taken to be a constant, $E_1 = \frac{u_o^2}{2}$ and $E_o(\phi) = \frac{(u_o + \frac{\phi^2}{2})^2}{2}$. The equation itself is an Abel's equation, and its solution is easily determined to be

$$\frac{d\psi}{dE} = \frac{2}{\pi} \frac{1}{\sqrt{u_o^2 - 2E}} . \tag{6.26}$$

Substituting this result in equation (6.20) for p gives

$$p = \frac{1}{2} \pm \frac{2}{\pi} \arctan \sqrt{\frac{2E - u_o^2 - u_o\phi^2 - \phi^4/4}{u_o^2 - 2E}} . \tag{6.27}$$

The resulting motions are shown in Figures 2 and 3.

The meridional motions show Hadley cells, as observed in the tropical regions of the earth's atmosphere. These cells extend to latitude $\hat{\phi} = \pm 2(Ro|u_o|)^{1/2}$ in either hemisphere. If we assume that the appropriate value of u_o is that which gives the global maximum value of $\hat{u}$ at the edge of the cell, we have $u_o = -1$, and we find that the cells extend to about 21^o north or

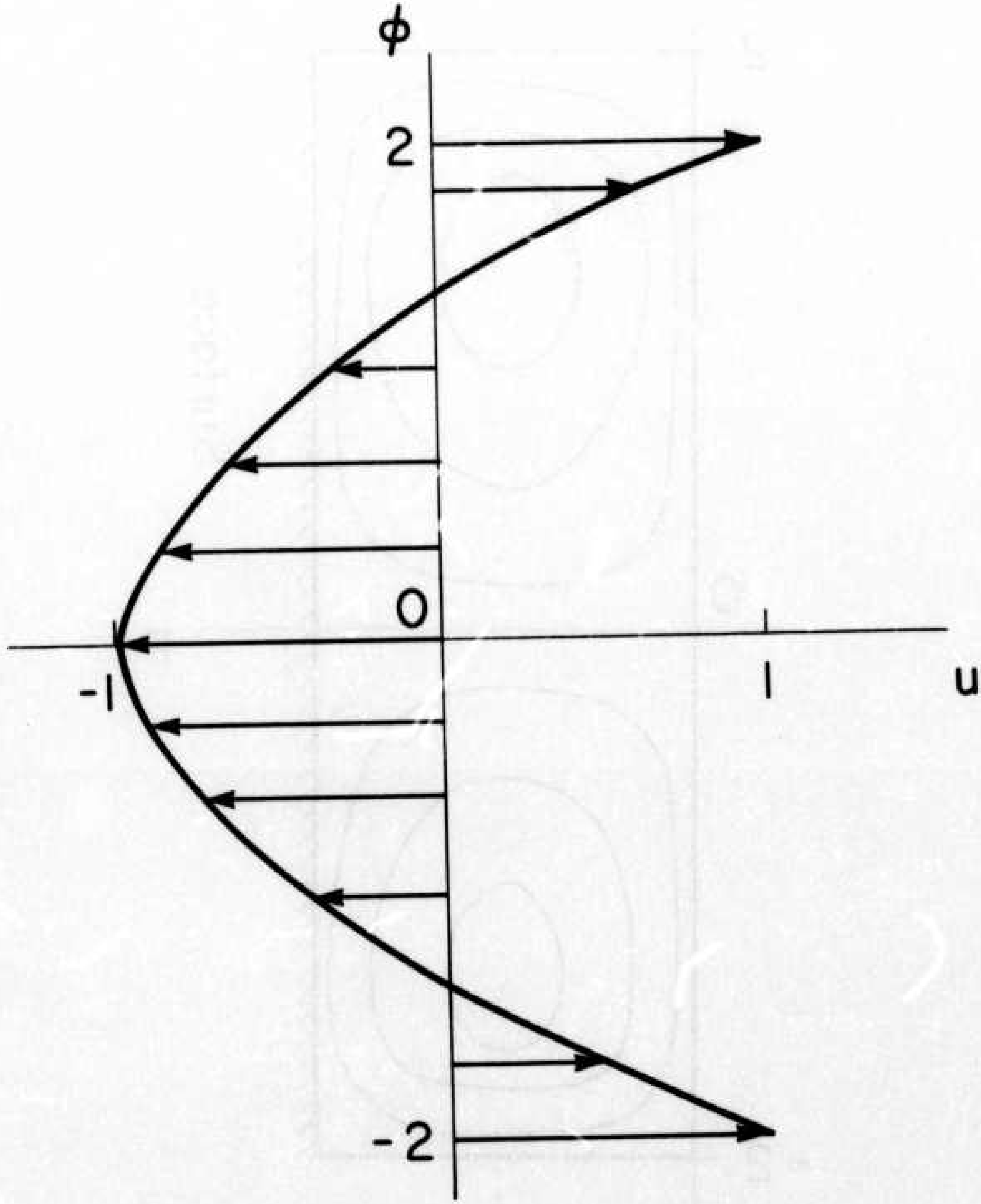

Figure 2. Zonal wind versus latitude (both scaled) in the equatorial cells.

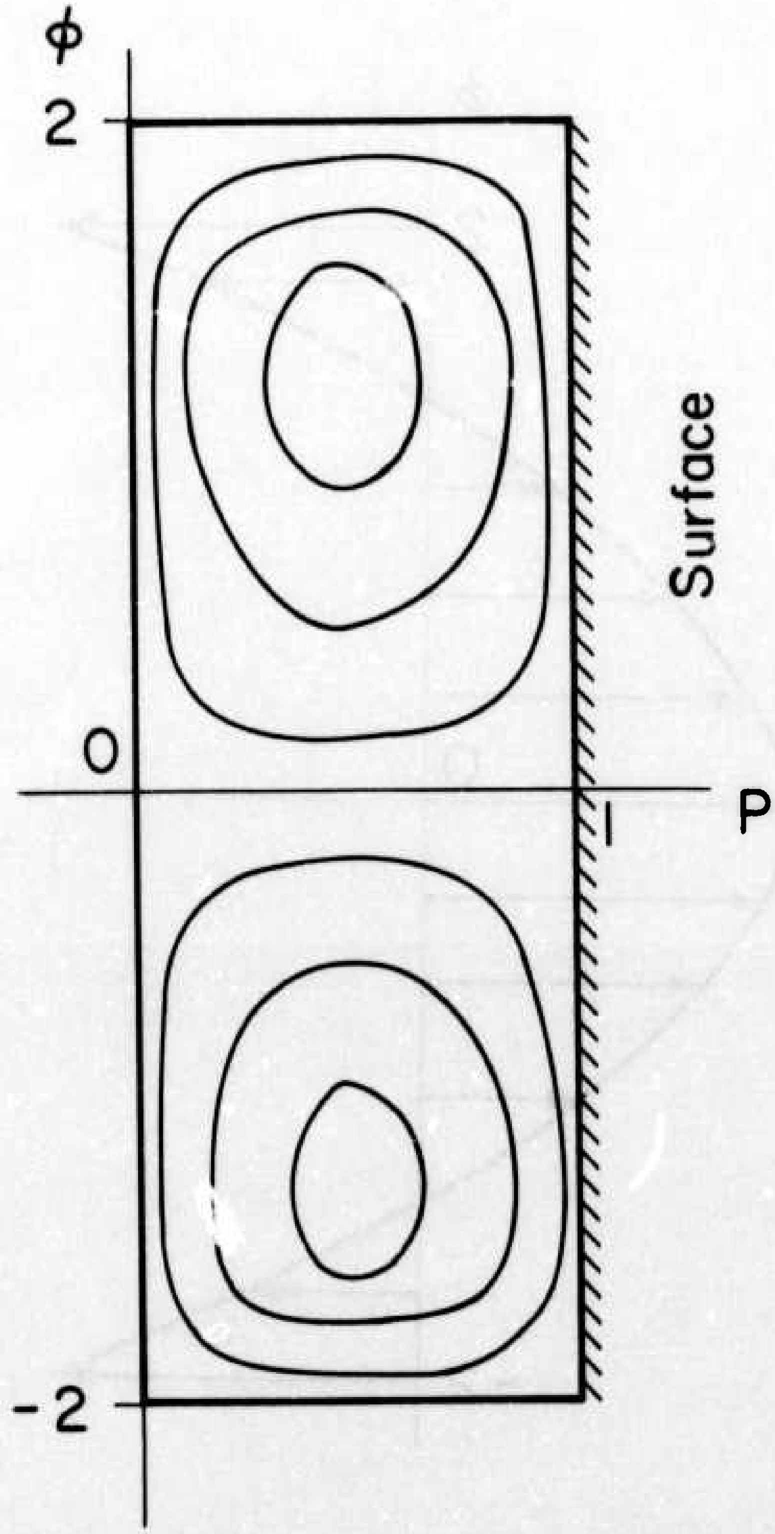

Figure 3. Streamlines for the meridional motion in scaled variables.

south latitude. With this assumption, the change from westerly winds to trade winds occurs at $\hat{\phi} = \pm \sqrt{2}\,(Ro|u_o|)^{1/2}$, corresponding to about 15^o. The predicted values agree reasonably well with observations.

Using considerations similar to those in Section 5, we find that the surface boundary layer in the tropical Hadley cell is governed by the equations

$$\frac{\partial}{\partial p}\left[\left(\frac{\partial u}{\partial p}\right)^2\right] = -\phi v + v\,\frac{\partial u}{\partial \phi} + \omega\,\frac{\partial u}{\partial p}\,, \tag{6.28}$$

$$\mu_V^2\,\frac{\partial}{\partial p}\left[\left(\frac{\partial u}{\partial p}\right)^2\right] = \phi u + v\,\frac{\partial v}{\partial \phi} + \omega\,\frac{\partial v}{\partial p}\,, \tag{6.29}$$

$$\frac{\partial v}{\partial \phi} + \frac{\partial \omega}{\partial p} = 0\,. \tag{6.30}$$

The vertical length scale for this layer is $(\rho_o g L_V^2 \frac{U}{2\Omega}(Ro)^{1/2})^{1/3}$. We shall not discuss this layer further here.

7. The Turbulent Connecting Layer

From equations (6.17) and (6.18) we see that the solutions in the Hadley cell region are defined only for $|\hat{\phi}| \leq 2(Ro|u_o(E_1)|)^{1/2}$, even when u_o is not assumed to be a constant. It is this failure of the solution which indicates that a turbulent boundary layer is needed to connect the Hadley cell to the temperate region. Moreover, since equations (6.8 - 6.10) have no forcing term, the driving mechanism for the Hadley cell must be movement of part of its boundary. We claim that the

Hadley circulation is driven by the turbulent layer at the poleward edges of the cells.

The correct scaling for the turbulent layer between the tropical Hadley cell and the temperate region is $\phi_c = 2(Ro)^{1/2}$ and $\phi_o = \varepsilon$. We assume that $\varepsilon \leq (Ro)^{1/2}$. The approximate equations become

$$\frac{\partial}{\partial\phi}\{(\frac{\partial u}{\partial\phi})^2\} = v\frac{\partial u}{\partial\phi} + \omega\frac{\partial u}{\partial p}, \tag{7.1}$$

$$\mu_H^2 \frac{\partial}{\partial\phi}(\frac{\partial u}{\partial\phi})^2 = v\frac{\partial v}{\partial\phi} + \omega\frac{\partial v}{\partial p}, \tag{7.2}$$

$$\frac{\partial v}{\partial\phi} + \frac{\partial\omega}{\partial p} = 0 \tag{7.3}$$

to order $(Ro)^{1/2}$.

These equations are difficult to solve. We shall satisfy ourselves by assuming that u is continuous, and noting that if we integrate (7.1) or (7.2) from 0 to 1 in p, and from $-\infty$ to ∞ in ϕ, we derive the result that across the boundary layer, the quantity

$$\int_0^1 (\frac{\partial u}{\partial\phi})^2 \, dp$$

is conserved. Physically, the second condition states that the total northward turbulent flux of eastward momentum through the layer is a constant.

We can estimate the thickness of the turbulent layer by noting that its latitudinal thickness scale is ε, the ratio of the eddy size to the earth's radius. If we assume an eddy size of a few hundred kilometers, then ε is comparable to Ro, the Rossby number, and ϕ_o is about 2^o. Thus we might expect its influence to be felt for about 10^o or so.

8. The Polar Region

The mechanics in the polar region is somewhat more complicated than that of the tropical regions. If we attempt a solution analogous to the tropical advective boundary layer surrounded by turbulent layers, the turbulent layers are as large as the advective region. This suggests that the equations appropriate in the polar region entail a balance between rotation, advection and turbulence. In order to model this, we assume that $\phi_c = \frac{\pi}{2}$, ϕ_o = Ro, $\alpha = 1$ and ε = Ro. The resulting approximate equations are

$$(8.1) \qquad v\frac{\partial u}{\partial \phi} + \omega \frac{\partial u}{\partial p} = v - \frac{uv}{\phi} - \frac{\lambda_H^2}{\phi}\frac{\partial}{\partial \phi}\{\phi(\frac{\partial u}{\partial \phi})^2\} ,$$

$$(8.2) \qquad v\frac{\partial v}{\partial \phi} + \omega \frac{\partial v}{\partial p} = -u + \frac{u^2}{\phi} + \frac{\mu_H^2}{\phi}\frac{\partial}{\partial \phi}\{\phi(\frac{\partial u}{\partial \phi})^2\} - \frac{\partial \Phi}{\partial \phi}(\frac{\pi}{2}, p) ,$$

$$(8.3) \qquad \frac{\partial v}{\partial \phi} + \frac{\partial \omega}{\partial p} + \frac{v}{\phi} = 0$$

to order Ro.

In terms of the polar dimensionless variables, the region under consideration is $0 \geq \phi$, $0 \leq p \leq p_n$ where $p_n = p_s(\frac{\pi}{2})/p_s(0)$.

In order that the geopotential be smooth at the pole, we must require $\partial\Phi(\frac{\pi}{2}, p)/\partial\phi = 0$.

The equations valid in the polar region are not readily treatable by classical means. We can, however, estimate the extent of this polar region, which we assume to be a cell, as in the tropical case. Since $\phi_0 = Ro$, corresponding to a few degrees latitude, the polar region should be several, perhaps 10 degrees wide. This is somewhat smaller than observations indicate for the earth.

9. The Energy Budget

A description of the motions of the atmosphere would not be complete without an explanation of the energy budget. The asymptotic approach taken here suggests that the following process is responsible for the dissipation of the potential energy caused by solar heating. That potential energy is in the form of a north-south geopotential gradient. This geopotential gradient drives the eastward flow in the middle latitudes, and hence is changed to the kinetic energy of the mean zonal flow there. Through turbulence generated by the zonal flow in the form of large scale eddies and their related vertical overturnings, the kinetic energy of the zonal flow is changed into kinetic energy of the meridional flow.

The zonal flow in the middle latitudes also drives large scale eddies at the interface between the tropical Hadley cell and the temperate region. The large scale eddies in turn drive the motions in the Hadley cell, where the zonal motion is coupled to the meridional motion through the Coriolis forces.

The zonal flow also drives the motions in the polar cells, which in general are weaker than those in the tropical cell. For a schematic representation of the energy cycle, see Figure 4.

10. Jupiter's Atmosphere

It is of interest to apply some of the ideas presented in this paper to the motions of the atmosphere of Jupiter. Jupiter's atmosphere, with the major exception of the Great Red Spot, appears to be zonally symmetric. In addition, the estimated Ekman and Rossby numbers for Jupiter are 10^{-8} and 10^{-2} respectively. In that case, the analysis presented here should be able to predict some features of the motion, provided that the atmosphere meets the other requirements herein, such as the shallow atmosphere assumption.

Since a knowledge of the geopotential is required for prediction of the winds in the temperate region, we shall not concern ourselves with that part of the atmosphere. Some features of the motions in the tropical zones, on the other hand, can be discerned from the value of the Rossby number. In particular, the motions show a Hadley cell, of width approximately 10°. This width agrees extremely well with the width of the equatorial zone on Jupiter. The present model also predicts trade winds, changing to prevailing westerlies at about 8°. This prediction is not observed; in fact the east-west circulation is observed to be eastward.

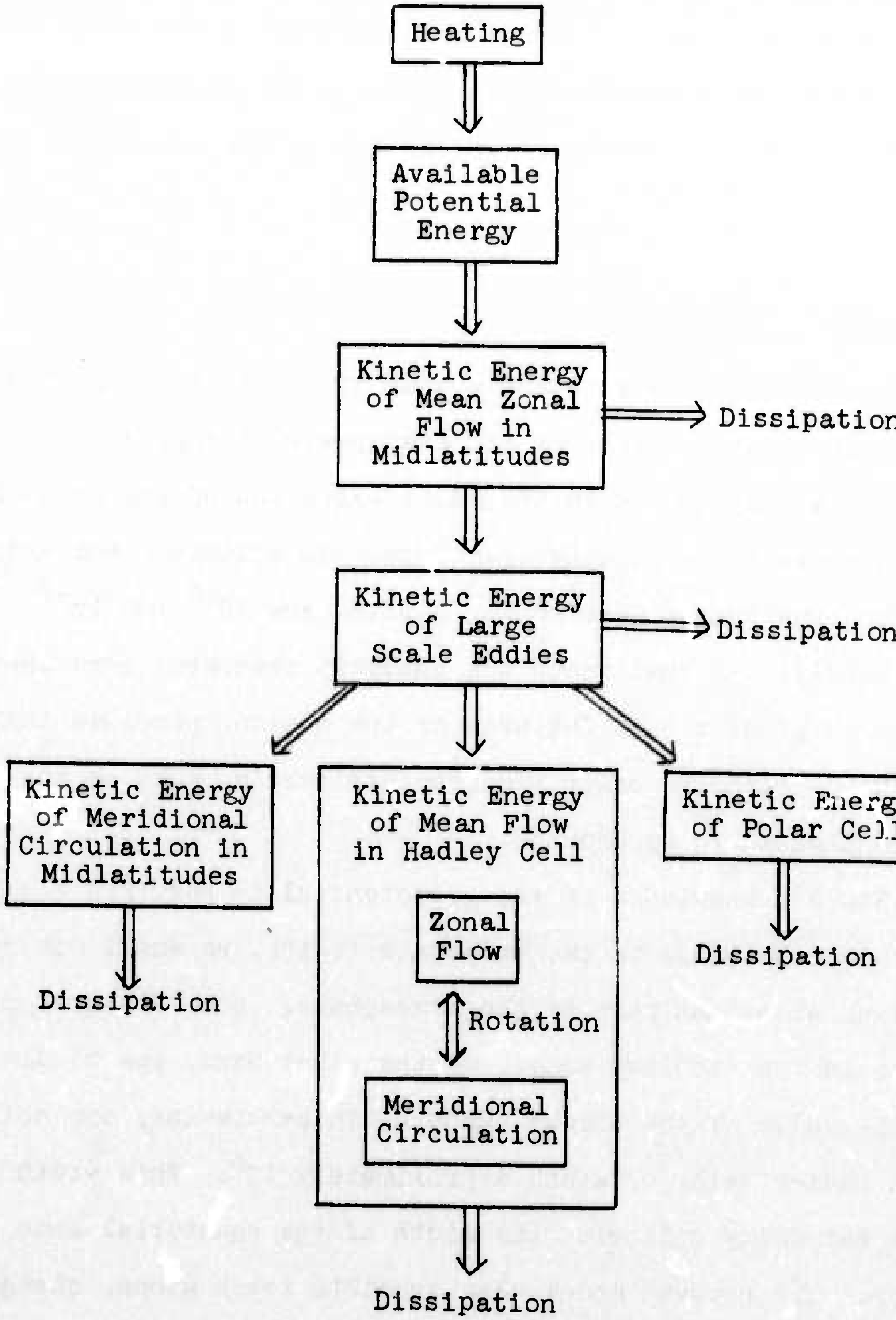

Figure 4. The driving mechanisms for global atmospheric motions.

11. Summary

Let us summarize the asymptotic scheme we have presented for approximating the motions of the earth's atmosphere. We separated thermodynamic considerations from the equations of motion by assuming that the temperature field is known. This allowed us to treat the fluid dynamical equations as forced by the geopotential.

We assumed a mixing length theory for the turbulence terms. By assuming that the eddy size is small compared to the earth's radius, we found that the effect of turbulence is confined to relatively small boundary layers.

We also assumed that the motions we seek are slow compared to the speed of the earth's rotation speed. This implies that the inertial terms are negligible except in layers which are relatively narrow compared to the whole atmosphere, but large compared to the turbulent layers. These advective boundary layers occur near the equator and the poles, and take the form of meridional cells.

The dynamics of the motion can be summarized as follows. In the temperate region, the north-south geopotential gradient drives the prevailing westerlies. Turbulent eddies drive the meridional motions. The eastward flow at the temperate edges of the polar and tropical cells drives the motions in these cells. In the equatorial cell, the balance between the Coriolis force and the advection causes the prevailing westerlies to change to trade winds near the equator, and also creates the meridional cell pattern.

There is a turbulent layer connecting the equatorial cell to the temperate zone.

The mechanics in the polar region is complicated by the fact that the turbulent terms, the Coriolis terms and the advective terms are all of equal importance.

Approximate solutions for the motion are not found in regions where turbulence is important. Presumably numerical techniques are needed for these regions.

To have a complete description of the atmospheric dynamics, we need to be able to derive expressions for geopotential $\hat{\Phi}$ and the density $\hat{\rho}$, which can be expressed in terms of the temperature $\hat{T}$ as $\hat{\rho} = \hat{p}/R\hat{T}$. The geopotential and temperature are determined through the energy equation and the vertical momentum equation, along with other balance laws for quantities which are important in thermodynamic considerations, such as water vapor, cloud cover, etc. In general, these equations involve the velocity components, for which we have approximate expressions involving the geopotential and the temperature. In theory, we need only substitute the appropriate expressions for the velocity components where they appear to yield a thermodynamic system involving only thermodynamic variables, independent of the velocity components. In fact, however, the expressions for the velocity components are not simple expressions involving the thermodynamic variables, and so this procedure does not lead to easily tractible thermodynamic equations. We shall leave the thermodynamic considerations for the future.

An approximate description of atmospheric dynamics could be useful in predicting climate changes due to various effects, for example, changes in cloud cover, solar absorption, etc. Most of these changes alter the thermodynamics of the atmosphere. In addition, the thermodynamics is affected by the motion of the atmosphere. What we have presented is an approximate description of the motions derived from given thermodynamic conditions. This reduces the problem to consideration of thermodynamic equations. Thus it is felt that this work represents a step toward a useful analytical approach to global climatology.

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